

Ricci Flow of Biaxial Bianchi IX Metrics

Gustav Holzegel^{a*}, Thomas Schmelzer^{b†} and Claude Warnick^{a‡}

^a*University of Cambridge, DAMTP, Wilberforce Road, CB1 0WA, Cambridge, U.K.*

^b*University of Oxford, Computing Laboratory, Wolfson Building, Parks Road, OX1 3QD, Oxford, U.K.*

12th June, 2007

Abstract

We use the Ricci flow with surgery to study four-dimensional $SU(2) \times U(1)$ -symmetric metrics on a manifold with fixed boundary given by a squashed 3-sphere. Depending on the initial metric we show that the flow converges to either the Taub-Bolt or the Taub-NUT metric, the latter case potentially requiring surgery at some point in the evolution. The Ricci flow allows us to explore the Euclidean action landscape within this symmetry class. This work extends the recent work of Headrick and Wiseman [8] to more interesting topologies.

1 Introduction

Ricci flow is a geometric flow within the space of Riemannian metrics on a given n -dimensional manifold \mathcal{M} defined by:

$$\begin{aligned} \frac{d}{dt}g(t, x) &= -2Ric_{g(t, x)} \\ g(0, x) &= g_0(x) \end{aligned} \tag{1.1}$$

where $g(t)$ is the metric at flow time t and $Ric_{g(t)}$ is its associated Ricci tensor. On a closed manifold, the PDE (1.1) admits short time existence [6], but will generally exhibit finite time blow-up. Work of Hamilton and Perelman has provided a means of continuing the flow beyond such a singularity, namely Ricci flow with surgery. There has recently been much interest in the Ricci flow with surgery, especially since the publication by Perelman of a proof of the Poincaré conjecture based on this method [10, 11].

*G.Holzegel@damtp.cam.ac.uk

†Thomas.Schmelzer@balliol.ox.ac.uk

‡C.M.Warnick@damtp.cam.ac.uk

For many practical purposes it is more convenient to consider a slightly modified Ricci flow, defined by:

$$\begin{aligned}\frac{d}{dt}g(t, x) &= -2Ric_{g(t, x)} + 2(\nabla\xi(t))_S \\ g(0, x) &= g_0(x)\end{aligned}\tag{1.2}$$

Here $\xi(t)$ is an arbitrary vector field on \mathcal{M} and $(\cdot)_S$ indicates that the tensor enclosed should be symmetrized on all its indices. The flows (1.1) and (1.2) are equivalent if we consider the flow to act on the space of metrics *modulo diffeomorphisms* as the second term in (1.2) on its own would define a flow through the equivalence class of a metric under diffeomorphisms. The advantage of (1.2) however is that one can pick the gauge $\xi(t)$, i.e. a diffeomorphism at each point of time, such that the equations become strongly parabolic. Short-time existence then follows from standard results [1, 2]. Moreover the stability of numerical codes is improved in this formulation of the Ricci flow.

In this paper we are going to study the Ricci flow on a class of four-dimensional manifolds \mathcal{M} whose boundary is given by a squashed 3-sphere. The physical motivation to study infilling metrics for a manifold with fixed boundary metric derives from an attempt to understand the thermodynamics of general relativity. In the Euclidean approach to the problem the metric is analytically continued to the Riemannian sector and – in analogy with statistical mechanics – a partition function associated with the canonical ensemble is defined by

$$\mathcal{Z}(g) = \int_{\mathcal{M}} d[g] e^{-S[g]}.\tag{1.3}$$

Here $S[g]$ is the Euclidean action (2.4) and the integration is performed over all metrics on \mathcal{M} admitting the same fixed boundary metric but (perhaps topologically) different interiors. The fixed boundary metric can be understood in a thermodynamic sense as holding the system at fixed temperature. However, the integral (1.3) is not well-defined in general and in the one-loop semi-classical approximation renormalization schemes have to be employed to extract the correct values for the entropy and other thermodynamic quantities from the partition function [5]. In practical evaluations of (1.3) one usually restricts to certain symmetry classes and invokes a saddle point approximation: the main contributions to the integral (1.3) will come from extremal points of the action, which are found to be Ricci-flat metrics. In this setting it is an interesting mathematical problem on its own to classify the number of infilling Ricci-flat metrics for a given boundary metric. This question has so far been answered analytically only in restricted symmetry classes (like the one studied in this paper). On the other hand, being a gradient flow, the Ricci flow provides a useful tool to explore the action landscape of Riemannian metrics. For metrics less constrained by symmetry assumptions, the Ricci flow may be thought of as a relaxation technique for finding new Ricci-flat metrics.

We direct the reader interested in further details, especially on the relation of the Ricci-flow to the thermodynamics of black holes and the renormalization group in quantum field theory, to [8]. In that paper the authors study the canonical ensemble for gravity in a box, modelled by four-dimensional manifolds with a boundary of topology $S^1 \times S^2$.¹ Furthermore, they restrict to metrics admitting a $U(1) \times SO(3)$ action by isometry.² At high temperatures, i.e. a small enough size of the S^1 direction, they find three infilling Ricci-flat geometries, which correspond to saddle points of the action: two

¹The S^1 direction arises from the periodic time coordinate of the Lorentzian metric after analytic continuation.

²Note that symmetry in the initial data is preserved under the Ricci flow.

Schwarzschild black holes of different horizon areas (with topology $\overline{B^2} \times S^2$) and hot flat space (topology $S^1 \times \overline{B^3}$). Considering perturbations of the small Schwarzschild solution it is finally shown numerically how the Ricci-flow converges to either the large black hole or hot flat space, with the latter case enforcing a surgery at some point.

The results of the present paper provide a natural extension of the work of [8] to other topologies. Here the boundary will be given by a squashed S^3 and we will restrict attention to manifolds endowed with metrics admitting an $SU(2) \times U(1)$ symmetry, so called biaxial Bianchi IX metrics. The Ricci flat metrics, i.e. the fixed points of the flow (1.1), within this class are the one parameter families of Taub-Bolt and Taub-NUT metrics, which differ in their topology. Taub-Bolt is a metric on $\mathbb{C}P^2 \setminus \{\text{open ball}\}$, whereas Taub-NUT is defined on $\overline{B^4}$. We will discuss some properties of these metrics, which play a role in Euclidean quantum gravity [5], in section 2. We also establish that fixing the squashing parameter of the 3-sphere on the boundary in a certain range in fact singles out three infilling Ricci flat geometries from these families: Two Bolt- and one NUT- solution. We demonstrate, by finding a negative eigenvalue of the linearized operator, that one of these Bolt solutions is unstable under the Ricci-flow. The full non-linear evolution of the unstable solution is finally studied numerically. We find that the unstable Bolt metric will flow to either the second (stable) Bolt metric or the NUT metric, depending on how the metric is perturbed initially. Geometrically the perturbation corresponds to whether one slightly increases or decreases the area of the small Bolt minimal surface. The crucial and most interesting feature, however, is that the latter flow from the Bolt to the NUT solution requires surgery because the NUT and Bolt solutions have different topology, as mentioned above. Hence we will explain our method of controlling the curvature blow up and how the surgery is finally performed. The behaviour under perturbations just described might be expected from an analysis of the gravitational Euclidean action for the solutions as illustrated below.

This work, together with that of Headrick and Wiseman, provide concrete examples of the two types of surgery required by the Ricci flow in 4 dimensions as anticipated by Hamilton in [7], which locally look like

$$S^3 \times B^1 \rightarrow B^4 \times S^0 \quad \text{and} \quad S^2 \times B^2 \rightarrow B^3 \times S^1. \quad (1.4)$$

respectively. These surgeries are expected to be irreversible.

To our knowledge the question of existence and uniqueness for the Ricci flow on manifolds with fixed boundary metric has not been answered in any generality. The reason is that in the strongly parabolic formulation (1.2) one typically ends up with mixed Dirichlet and Neumann conditions at the boundary. We will therefore prove uniqueness of the evolution for our Ricci-flow equations directly in appendix A.

2 NUTs and Bolts

2.1 Regular Biaxial Bianchi IX Metrics

The biaxial Bianchi-IX metrics are a class of metrics admitting an $SU(2)_L \times U(1)_R$ symmetry group. They may be written in terms of the usual left-invariant $su(2)$ one-forms σ_i in the form:

$$ds^2 = a(r)^2 dr^2 + b(r)^2 (\sigma_1^2 + \sigma_2^2) + c(r)^2 \sigma_3^2 \quad (2.1)$$

The $SU(2)_L$ symmetry arises from the left action of $SU(2)$ under which the one-forms σ_i are by definition invariant. In addition, since σ_1 and σ_2 appear in the metric with the same coefficient there

is a right action by a $U(1)$ subgroup of $SU(2)$ which acts by isometries. We will be interested in manifolds with a boundary, so the radial coordinate r will take values in the range $r_0 < r \leq r_1$. The boundary at r_1 is topologically a 3-sphere and the induced metric that of a homogeneously squashed S^3 . We assume that the functions a, b, c are smooth and non zero on the interval and further that $\{r = r_0\}$ is a fixed point set of the $U(1)_R$ action. This means that $c(r_0) = 0$ and there will in general be a singularity at $r = r_0$ unless certain conditions are met. Without loss of generality, we assume that $r_0 = 0$ and $a(r)^2 \sim a_0^2 + O(r^2)$. If the metric is to be regular at $r = 0$ there are two possibilities [4]:

Case 1 $b(0) \neq 0$. In this case regularity requires that

$$c(r)^2 \sim \frac{1}{4}a_0^2 r^2 + O(r^4) \quad (2.2)$$

as $r \rightarrow 0$. A coordinate transformation then shows that $r = r_0$ is a coordinate singularity, similar to the origin of \mathbb{E}^2 in polar coordinates. The point set $r = 0$ is then a metric 2-sphere known as a *Bolt*. Geometrically it is a minimal surface rather than a boundary.

Case 2 $b(0) = 0$. In this case regularity requires that

$$b(r)^2 \sim c(r)^2 \sim \frac{1}{4}a_0^2 r^2 + O(r^4) \quad (2.3)$$

as $r \rightarrow r_0$. A coordinate transformation shows that $r = 0$ is a coordinate singularity, similar to the origin of \mathbb{E}^4 in spherical polar coordinates. The point set $r = 0$ is then a point known as a *Nut*.

2.2 Ricci Flat Metrics

Clearly from (1.1) the fixed points of the Ricci flow are metrics satisfying $Ric_g = 0$. Such metrics are sometimes known as gravitational instantons and occur as extremal points of the Euclidean Einstein-Hilbert action:

$$S[g] = -\frac{1}{16\pi G} \int_{\mathcal{M}} dV \sqrt{g} R - \frac{1}{8\pi G} \int_{\partial\mathcal{M}} dS \sqrt{h} K. \quad (2.4)$$

With a carefully chosen metric on the space of Riemannian metrics, the Ricci flow may be thought of as a gradient flow of this action, as discussed by Headrick and Wiseman [8].

2.2.1 Taub-Bolt

In order to find Ricci flat infilling metrics for a squashed S^3 boundary, we consider Ricci flat biaxial Bianchi IX metrics whose radial slices are squashed S^3 . There are two families of such metrics corresponding to the two possible regular structures at the fixed points of the $U(1)$ action. The Taub-Bolt family of metrics is given by:

$$ds^2 = \frac{r^2 - n^2}{r^2 - \frac{5}{2}nr + n^2} dr^2 + (r^2 - n^2) (\sigma_1^2 + \sigma_2^2) + 4n^2 \left(\frac{r^2 - \frac{5}{2}nr + n^2}{r^2 - n^2} \right) \sigma_3^2 \quad (2.5)$$

This metric is Ricci flat and regular for r in the range $2n < r < \infty$ and there is a bolt at $r = 2n$. In order that this metric may be considered as infilling a squashed S^3 we introduce a boundary by

restricting r to $2n < r \leq R$. It will be convenient to change the radial coordinate and introduce new parameters $\alpha = \sqrt{2 + \frac{3n}{R^2}}$, $\beta = \sqrt{3}n$ instead of n, R so that the metric instead takes the form:

$$ds^2 = 8\beta^2\alpha^2 \frac{(\alpha^4 - \rho^4)}{(\alpha^2 - 2\rho^2)^4} d\rho^2 + \beta^2 \frac{(\alpha^4 - \rho^4)}{(\alpha^2 - 2\rho^2)^2} (\sigma_1^2 + \sigma_2^2) + 2\beta^2\alpha^2 \frac{\rho^2}{\alpha^4 - \rho^4} \sigma_3^2 \quad (2.6)$$

Now the radial coordinate is in the range $0 < \rho \leq 1$. For $\alpha^2 > 2$ this is a regular metric on a manifold with boundary which is topologically \mathbb{CP}^2 with an open 4-ball removed. In the limit $\alpha^2 \rightarrow 2$ the boundary recedes to infinity and we have the asymptotically locally flat Taub-Bolt metric on $\mathbb{CP}^2 - \{\text{pt.}\}$.

We seek Ricci flat metrics which have a given metric on the boundary S^3 , taken to be:

$$h = \mu^2 (\sigma_1^2 + \sigma_2^2) + \nu^2 \sigma_3^2. \quad (2.7)$$

h can only be the induced metric on the boundary of a Taub-Bolt solution for certain values of $\tau^2 = \nu^2/\mu^2$, which we refer to as the squashing parameter. If $\tau^2 > 1 - \frac{9}{8}3^{\frac{1}{3}} + \frac{3}{8}3^{\frac{2}{3}} = \tau_c^2$ then one cannot find a Taub-Bolt solution whose boundary has metric (2.7). For $\tau^2 < \tau_c^2$ there are two choices of (α, β) such that the boundary metric is h . These can be distinguished via the parameter β which parameterises the size of the bolt. The two solutions are thus distinguished by having a ‘small bolt’ or ‘large bolt’. In the critical case where $\tau = \tau_c$, the small and large bolt solutions coincide and we have only one infilling Taub-Bolt geometry.

2.2.2 Taub-Nut

We now turn to the other family of Ricci flat biaxial Bianchi IX metrics which may be considered as infilling geometries for the boundary S^3 with metric (2.7). These contain a nut and are known as the Euclidean self-dual Taub-NUT solutions. In their usual form they have metric:

$$ds^2 = \frac{r+n}{r-n} dr^2 + 4n^2 \frac{r-n}{r+n} \sigma_3^2 + (r^2 - n^2) (\sigma_1^2 + \sigma_2^2). \quad (2.8)$$

This metric is regular and Ricci flat for r in the range $n < r < \infty$. At $r = n$ the metric has a regular nut and we introduce a finite boundary by restricting r to $n < r \leq R$. As in the previous case, it will be convenient to change to a different radial coordinate and new parameters $\gamma = \frac{1}{R}\sqrt{2n}$, $\delta = \sqrt{R}$ so that the metric takes the form:

$$ds^2 = 4\delta^2(\rho^2 + \gamma^2) d\rho^2 + \delta^2 \rho^2 \left((\rho^2 + \gamma^2)(\sigma_1^2 + \sigma_2^2) + \frac{\gamma^4}{\rho^2 + \gamma^2} \sigma_3^2 \right), \quad (2.9)$$

where the radial coordinate is in the range $0 < \rho \leq 1$. One can see by inspection that this metric satisfies the conditions for $\rho = 0$ to be a regular nut. This gives a regular metric on $\overline{B^4}$, the closed 4-ball. There is exactly one Taub-Nut metric which infills the squashed S^3 (2.7) for τ in the range $0 < \tau < 1$.

2.2.3 Action

As we know that the Ricci flow is a gradient flow of the Euclidean action (2.4), it is convenient to plot the action of the Ricci flat solutions. We fix the volume of the boundary metric and plot the difference

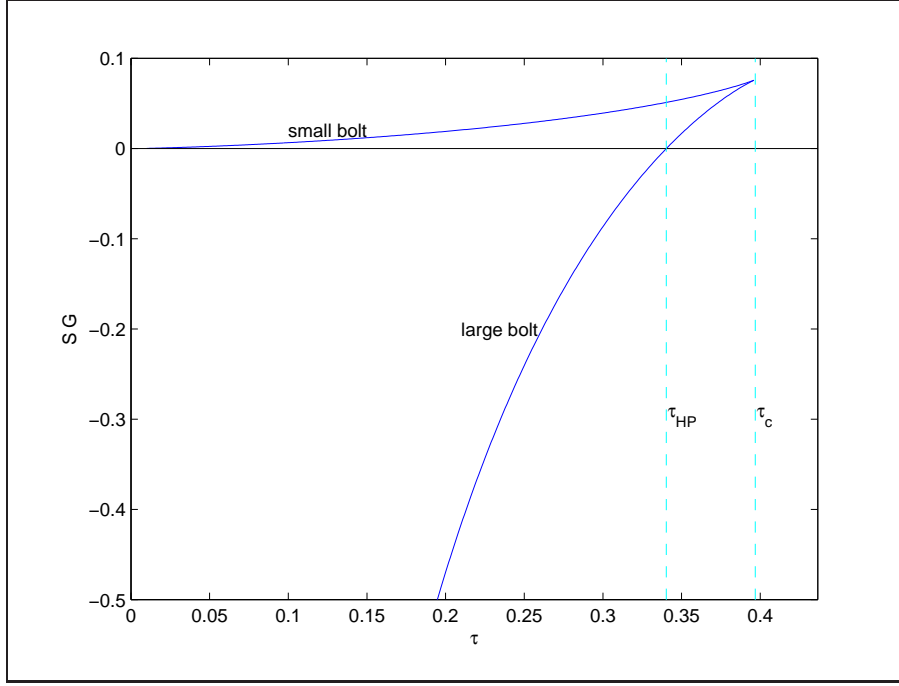


Figure 1: A plot of $S_{\text{Bolt}} - S_{\text{Nut}}$ as a function of boundary squashing

$S_{\text{Bolt}} - S_{\text{Nut}}$ against τ for both branches of the infilling Taub-Bolt solutions. This plot is included as Figure 1. We note that the solution with the greatest action is always the small bolt solution. The solution with the least action is either the large bolt or the nut, according as $\tau < \tau_{HP}$ or $\tau > \tau_{HP}$. At $\tau = \tau_{HP}$ there is a Hawking-Page type transition as the global minimizer of the action changes from one solution to the other.

In many ways, the Taub-Bolt geometries are similar to the Euclidean Schwarzschild geometries considered in [8]. After analytic continuation, the horizon of the Schwarzschild black hole becomes a regular bolt provided the Wick rotated time coordinate has some suitable period. If we seek geometries which infill an $S_R^2 \times S_\beta^1$ boundary, then we find that this can only be done with a Euclidean Schwarzschild metric for a particular range of the ratio β/R . As with Taub-Bolt there is a critical value for this ratio where there is a bifurcation and two solutions appear. The Schwarzschild solution with a small bolt has greatest action and is unstable under the Ricci flow, flowing to either the large bolt solution or to periodically identified flat space.

Motivated by this analogy to Euclidean Schwarzschild, we expect the small bolt solution to be unstable under Ricci flow in the case of biaxial Bianchi IX symmetry. More precisely, perturbing the small bolt solution the metric should flow to either the large bolt solution or the Taub-Nut solution, the latter flow requiring a topology changing surgery. We will show in the remainder of the paper that the flow indeed exhibits the behaviour described.

3 Ricci flow simulations

3.1 Bolt topology – the linearized problem

We will begin our simulations by perturbing the small bolt solution as we expect this to be unstable against the Ricci flow. For the flow on a manifold with bolt topology we make the metric ansatz:

$$ds^2 = e^{2A(r,t)} dr^2 + e^{2B(r,t)} (\sigma_1^2 + \sigma_2^2) + r^2 e^{2C(r,t)} \sigma_3^2 \quad (3.1)$$

with $0 < r \leq 1$. The Taub-Bolt solutions may be cast into this form. We also make a choice of ξ , the vector field which specifies the gauge in (1.2). We take the usual (θ, ϕ, ψ) coordinates for $SU(2)$ and set

$$\xi = [-(g_{\mu\nu} \star d \star dx^\nu) + (g_{\mu\nu} \star d \star dx^\nu)_0] dx^\mu = -(2B' + C' - A') dr, \quad (3.2)$$

where the subscript 0 indicates that the term should be evaluated on some fixed reference metric, which we take to be (2.6). One can check that ξ vanishes identically in these coordinates. These are the coordinates singled out by the gauge choice, which breaks the diffeomorphism invariance of the Ricci flow equations. This choice of gauge, similar to that chosen for the ‘deTurck trick’ [2] makes the Ricci flow equations strongly parabolic. We wish to evolve the functions on the interval $0 < r \leq 1$, however $r = 0$ is not a boundary for the manifold, but a point at which the coordinates break down. This is reflected in the equations we derive from (3.1, 3.2) as some coefficients of the PDE are not bounded as $r \rightarrow 0$, cf. the system (A.2)-(A.4) in the appendix. The PDE is expected to admit a well posed initial value formulation, provided that we impose that the functions A, B, C extend smoothly to even functions on the interval $-1 \leq r \leq 1$ as will be discussed in appendix A, where we indeed prove the uniqueness of solutions with given initial and boundary data. Doubling the interval is therefore a convenient strategy to remove the apparent boundary at $r = 0$. The pseudo-spectral collocation methods [3] we have used for the numerical simulations benefit to a large extent from this idea. We now need three boundary conditions at $r = 1$.³ Two of these arise by fixing the metric on the boundary. For well posedness we require a third, which comes from requiring the vector field ξ to vanish at the boundary. This ensures that the boundary remains at $r = 1$. The boundary conditions are then:

$$\begin{aligned} B(1, t) &= B(1, 0) \\ C(1, t) &= C(1, 0) \\ \xi(1, t) &= 0 \end{aligned} \quad (3.3)$$

We expect that the small bolt solution will be unstable under this flow and that the large bolt solution will be stable. In order to check this, we linearize the system of equations obtained above about the known Taub-Bolt solution and seek solutions of the form

$$A(r, t) = A_0(r) + e^{-\lambda t} a(r), \quad B(r, t) = B_0(r) + e^{-\lambda t} b(r), \quad C(r, t) = C_0(r) + e^{-\lambda t} c(r), \quad (3.4)$$

where a, b, c are considered small. It can be shown that λ is then an eigenvalue of the Lichnerowicz Laplacian. If the Lichnerowicz Laplacian has a negative eigenmode, the perturbations will grow

³those for $r = -1$ follow by symmetry

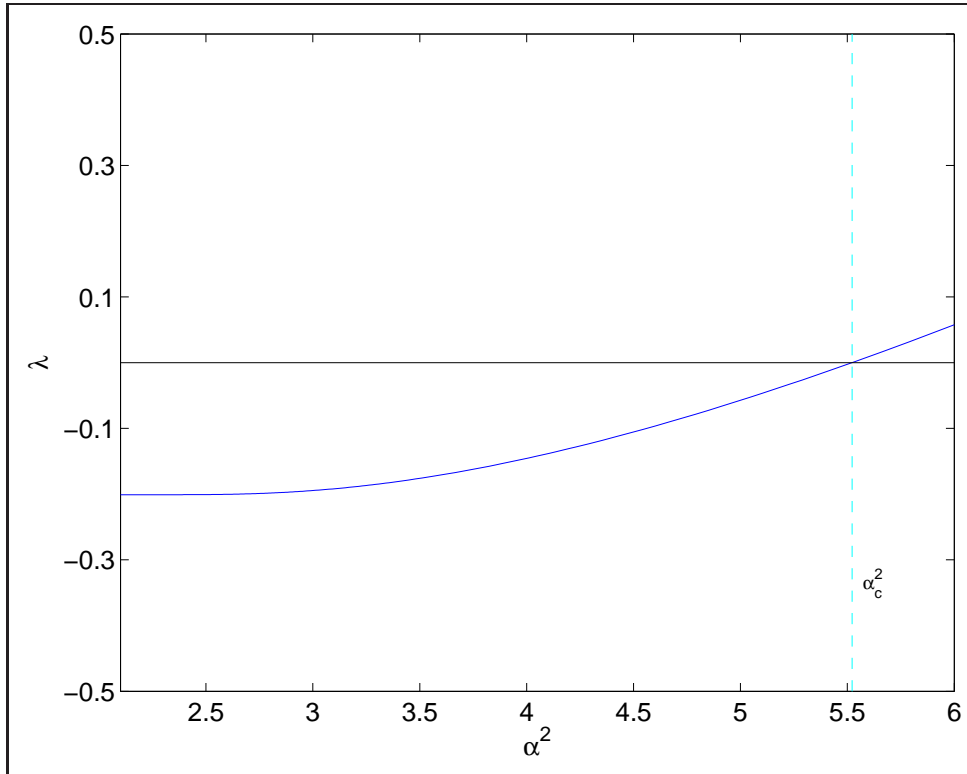


Figure 2: A plot of λ_{min} as a function of α

without bound and the solution will be unstable. We have calculated numerically, using a pseudo-spectral method based on collocation in Chebyshev points [13, Chap. 9] the lowest eigenvalue of the Lichnerowicz Laplacian, in the gauge ξ . It is plotted in Figure (2) as a function of α , for $\beta^2 = 3$. We find, as expected, that the lowest eigenvalue is negative for $\alpha^2 < \alpha_c^2$ which is the range of parameters corresponding to the small bolt. The eigenvalue vanishes at the critical point where the small and large bolt coincide and becomes positive for the large bolt. With 38 sampling points, the change in sign occurs at the theoretical value of α_c^2 to within 10^{-12} . Furthermore, as $\alpha \rightarrow 2$, the boundary approaches infinity and we recover the result of Young [15] that ALE Taub-Bolt has a negative Lichnerowicz mode with $\lambda \in (-0.200, -0.201)$.⁴ The choice of β above is so as to make a direct comparison with Young's work possible. See also [14] where a cosmological constant is included. Such negative modes may be interpreted as indicating a semi-classical instability of the metric considered as a gravitational instanton [5]. They also indicate a dynamical instability within general relativity of the 5-dimensional ultra-static vacuum metric obtained by adding a $-d\tau^2$ term to the metric.

⁴For values of α^2 near 2, one must increase the number of sampling points to obtain accurate results.

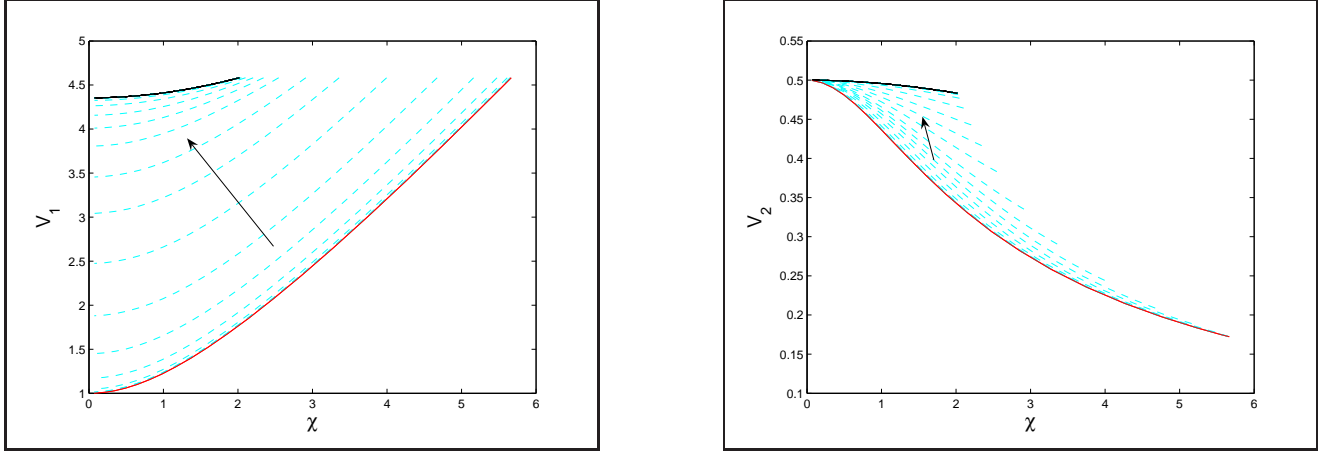


Figure 3: Plots of $V_1(\chi)$ and $V_2(\chi)$ at selected times during the flow for $\beta = 1, \alpha^2 = 2.5$. The flow proceeds in the direction of the arrow. The two infilling bolt geometries are shown as solid lines.

3.2 Taub-Bolt Topology – the Initial Flow

We use the solutions to the linearised problem to seed the flow for the full non-linear equations. We consider the flow with initial conditions:

$$A(r) = A_0(r) + \epsilon a_-(r), \quad B(r) = B_0(r) + \epsilon b_-(r), \quad C(r) = C_0(r) + \epsilon c_-(r) \quad (3.5)$$

Where the functions a_- etc. are the negative mode found for the linearised problem, normalised so that $b_-(0) = 1$. In this case a perturbation with $\epsilon > 0$ increases the area of the bolt, while one with $\epsilon < 0$ decreases it.

We first consider perturbing the small bolt solution with $\epsilon > 0$ which produces a small increase in the area of the bolt. The PDEs were again discretized in space using pseudo-spectral collocation in Chebyshev points. The time-stepping was performed with MATLAB's `ode15s` numerical integrator. We find that for a range of values of α^2 and $\epsilon > 0$ the metric at time t approaches the large bolt solution as t becomes large. In order to plot the results, we first cast the metric in the gauge invariant form:

$$ds^2 = d\chi^2 + V_1(\chi)^2 (\sigma_1^2 + \sigma_2^2) + \chi^2 V_2(\chi)^2 \sigma_3^2 \quad (3.6)$$

and then plot the functions V_1 and V_2 . We see that the flow starting at the small bolt converges to the large bolt. Taking 102 spatial sampling points and $\beta = 1, \alpha^2 = 2.5$, we find that for $t > 40$ the functions have uniformly approached the large bolt solution to within 10^{-9} , with better accuracy for more sampling points.⁵ For larger values of α^2 we also get better accuracy, indeed for $\alpha^2 = 4$ we can achieve 10^{-11} accuracy with only 42 sampling points.

We now turn to the situation where $\epsilon < 0$. In this case, the numerical simulations appear to exhibit a finite time blow-up which occurs because the bolt becomes smaller and smaller and approaches zero size in finite time. This behaviour where an S^2 collapses to a point in finite time is characteristic of the Ricci flow. In order to continue the flow, we must perform a surgery, turning the bolt at the origin to a nut. To decide when such a surgery should be performed, we monitor the value of the coordinate invariant scalar $K = \text{Riem}^2|_{r=0}$. We stop the flow when $t = t_c$, defined by $K(t_c) > 10^8$.

⁵However, there is a limit due to the fact that for very large N the matrices typically become ill-conditioned.

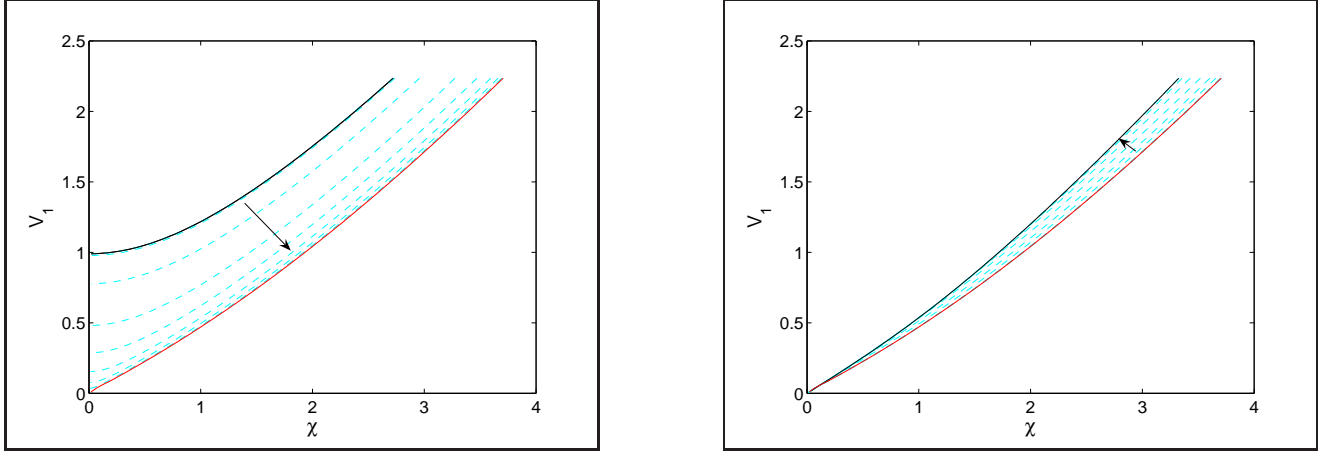


Figure 4: Plots of $V_1(\chi)$ at various times, both before surgery (on the left) and after surgery (on the right). The solid red line shows the function after surgery on both plots. The solid black lines are the functions for Taub-Bolt on the right and Taub-NUT on the left.

3.3 Taub-NUT topology – Surgery and Beyond

The idea behind surgery for the Ricci flow is that given a solution to the Ricci flow equations exhibiting finite time blow-up at a time T , we should take the Riemannian manifold at a time $T - \delta$ and remove a small region of size ϵ from the manifold and glue in a new piece which has a different topology and smoothly extends the metric on the rest of the manifold. We then re-start the Ricci flow. If this is done appropriately, the resulting flow should not depend on the details of the gluing in the limit $\delta \rightarrow 0$.

In our case, the minimal S^2 at $r = 0$ appears to be shrinking to zero size in a finite time. In order to continue the flow beyond this singularity, we remove the region $r < \epsilon$ from the manifold and glue in a 4-ball. This means that the new metric will take the form:

$$ds^2 = e^{2\tilde{A}} \left(dr^2 + \frac{r^2}{4} e^{-2r^2\tilde{C}} \left(e^{2r^2\tilde{B}} (\sigma_1^2 + \sigma_2^2) + \sigma_3^2 \right) \right) \quad (3.7)$$

We require that this agrees with the metric at the point we stopped the flow, $g(t_c)$, for $r > \epsilon$ and is everywhere smooth. This leaves considerable ambiguity in prescribing the metric functions for $r < \epsilon$, however numerical experimentation showed that the long term properties of the flow are insensitive to the precise gluing technique.

We now wish to continue the Ricci flow with the new initial conditions given by the metric functions after surgery. We will again introduce a gauge field $\tilde{\xi}$ to make the Ricci flow equations strongly parabolic. We make a similar choice of gauge, given by:

$$\tilde{\xi} = [-(g_{\mu\nu} \star d \star dx^\nu) + (g_{\mu\nu} \star d \star dx^\nu)_0] dx^\mu = - \left[2\tilde{A}' + 2(r^2\tilde{B})' - 3(r^2\tilde{C})' \right] dr. \quad (3.8)$$

This time we use the Taub-NUT metric in coordinates (2.9) as a reference metric. The equations which arise are once again symmetric under $r \rightarrow -r$ and so we can set the boundary conditions at the origin by requiring that the functions $\tilde{A}, \tilde{B}, \tilde{C}$ extend smoothly to an even function on $[-1, 1]$. The

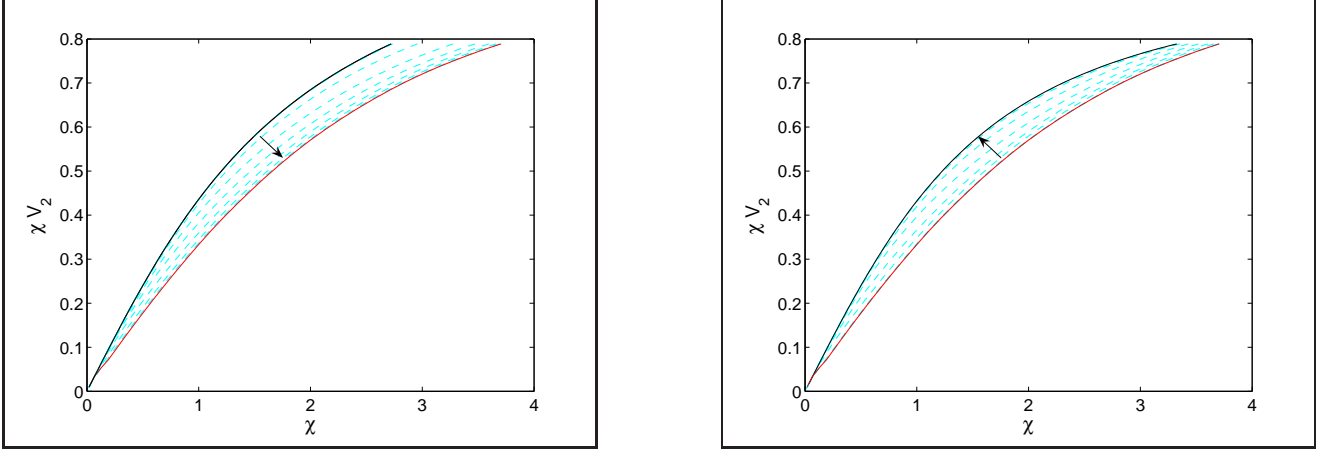


Figure 5: Plots of $\chi V_2(\chi)$ at various times, both before surgery (on the left) and after surgery (on the right). The solid red line shows the function after surgery on both plots. The solid black lines are the functions for Taub-Bolt on the right and Taub-NUT on the left.

symmetric boundary conditions are then given by:

$$\begin{aligned} (\tilde{A} - \tilde{C} + \tilde{B})(1, t) &= (\tilde{A} - \tilde{C} + \tilde{B})(1, 0) \\ (\tilde{A} - \tilde{C})(1, t) &= (\tilde{A} - \tilde{C})(1, 0) \\ \tilde{\xi}(1, t) &= 0 \end{aligned} \tag{3.9}$$

which indicate that the boundary metric is fixed and the diffeomorphism $\tilde{\xi}$ fixes $r = 1$. This final condition is not satisfied by the functions obtained via surgery, however a near identity change of the r coordinate can make the initial conditions compatible with the boundary conditions.

We can show (see Appendix) that the system of PDEs obtained from (1.2) with the gauge choice (3.8) and boundary conditions (3.9) has the uniqueness property. We believe that short term existence also holds, making the problem well posed. One might think that the parity condition could be replaced by $\tilde{A}'(0, t) = 0$ etc. however it appears that with this boundary condition the system is ill posed and short term solutions do not exist for generic initial data.

The Ricci flow equations were solved numerically both before and after surgery using the same technique as for the flow not requiring surgery, starting from a Taub-Bolt metric with $\beta = 1$, $\alpha^2 = 3.5$. The results are shown in figures 4 and 5. For these plots it is more convenient to plot V_1 and $\chi V_2(\chi)$, where the functions V_i are defined as in the previous section. We see that although the curvature blows up at $r = 0$ under the flow within the bolt topology, the gauge invariant functions are well behaved and the change due to the surgery is slight. We find that for sufficiently many sampling points the final functions approach the functions of Taub-NUT uniformly to within 10^{-6} . Experimentation with arbitrary initial conditions suggests that the Taub-NUT metric is the final state of the flow for any metric within the \bar{B}^4 topology with appropriately squashed S^3 boundary conditions.

4 Conclusion and Open Questions

In this paper we explored the Euclidean action landscape of biaxial Bianchi IX metrics using the Ricci flow with surgery. We demonstrated the existence of three saddle points of the Euclidean action and showed numerically how the Ricci-flow takes us from one to the other, namely from the unstable Bolt-solution to either the stable Bolt or the NUT-solution depending on the perturbation.

It is an ambitious analytical problem to make the observations of the present paper rigorous, which would involve applying the methods of Hamilton and Perelman to study the long-time behaviour of the Ricci-flow. We finish the paper with two conjectures, which we hope to address rigorously in the future. The first expresses our belief that we have exhausted the qualitative features of the Ricci-flow within this symmetry class by our study:

Conjecture 4.1. *For arbitrary biaxial Bianchi-IX symmetric initial-data and fixed metric on the boundary S^3 given by (2.7) with $0 < \tau < 1$, Ricci-flow plus at most one surgery will converge to a member of either the Taub-Bolt or the Taub-NUT family.*

Restricting to the subcase of NUT-topology, one might – encouraged by the final remark of the previous section – hope to prove the stronger

Conjecture 4.2. *For any biaxial Bianchi-IX symmetric initial-data with Taub-NUT topology $(\overline{B^4})$, the Ricci-flow admits long-time existence and converges to a member of the Taub-NUT family.*

It should be interesting to see how the different geometric features of the Bolt and the Nut make their way into the estimates necessary to establish the long-time behaviour of the flow.

Let us finally mention possible extensions of the present work. With the tools at hand it should be possible to include a cosmological constant into the formulae and impose AdS boundary conditions. We expect similar results to hold with Taub-Bolt and Taub-NUT being replaced by their AdS generalizations.

Moreover, there is a topological generalization if one considers the more general Lens spaces instead of the squashed S^3 to function as the fixed boundary metric. For the case of $\mathbb{R}P^3$ we expect the Eguchi-Hanson metric to be a stable infilling solution.

5 Acknowledgements

We would like to thank Mihalis Dafermos, Gary Gibbons, Christoph Ortner, Julian Sonner and David Tong for useful discussions. Moreover, we are grateful to the following funding bodies for financial support: Studienstiftung des deutschen Volkes and EPSRC (GH), PPARC (CMW) and the Rhodes Trust (TS).

A Uniqueness

In this appendix we are going to prove uniqueness of the solution to our modified Ricci-flow equations. As we noted in the introduction, short-time existence and uniqueness has been proven for the

Ricci flow in general on closed manifolds. On the other hand, for manifolds with boundary the only general existence and uniqueness result we know of is [12] where the Neumann problem

$$\begin{aligned}\partial_t g_{ij}(x, t) &= -2R_{ij}(x, t) \\ g_{ij}(x, 0) &= \hat{g}_{ij}(x) \\ K_{ab} &= 0 \text{ on } \partial\mathcal{M}^n\end{aligned}\tag{A.1}$$

with K_{ab} being the extrinsic curvature on the boundary, is studied. It turns out that the boundary has to satisfy certain geometric conditions in order for (A.1) to admit short-time existence: For a totally geodesic boundary existence and uniqueness can be shown [12] by an application of the general results on parabolic boundary value problems described in [9].

The Dirichlet problem, the problem of fixing the metric itself on the boundary, has not yet been considered in generality. The reason seems to be that the deTurck technique will always introduce a derivative condition at the boundary and the resulting boundary conditions for the modified flow are of non-standard type. Note that our boundary condition (3.3) is typical in that sense.

This motivates a more detailed study of existence and uniqueness within the biaxial symmetry class considered in the paper. Here we will prove uniqueness of the modified Ricci-flow. A more practical reason which originally initiated the study was the occurrence of numerical solutions blowing up near the origin in our evolution. It turns out that in addition to the boundary conditions a topological condition has to be imposed near the origin corresponding to the avoidance of a conical singularity. Without this additional condition our uniqueness proof would not go through.⁶

The Ricci-flow equations for the metric ansatz (3.1) constitute the following non-linear system

$$\dot{A} = e^{-2A}A'' - \frac{1}{r}e^{-2A}A' + \frac{2}{r}e^{-2A}C' + e^{-2A}(C'^2 + 2B'^2 - A'^2)\tag{A.2}$$

$$\dot{B} = e^{-2A} \cdot B'' + \frac{1}{r}e^{-2A}B' - e^{-2B} + \frac{1}{2}r^2e^{-4B+2C}\tag{A.3}$$

$$\dot{C} = e^{-2A} \cdot C'' + \frac{1}{r}e^{-2A}C' - \frac{1}{2}r^2e^{-4B+2C}\tag{A.4}$$

The variable r ranges from 0 to 1. We will prove uniqueness within the space of smooth *even* functions. In particular, the Neumann conditions, $A'(t, 0) = 0$, $B'(t, 0) = 0$, $C'(t, 0) = 0$ hold at $r = 0$. At $r = 1$ the Dirichlet conditions, $B(t, 1) = B_0(1)$, $C(t, 1) = C_0(1)$, and the Neumann condition, $(2B' + C' - A')(t, 1) = 0$ is imposed, as was outlined in section 2. Moreover, to avoid a conical singularity at the origin, we must have $\lim_{r \rightarrow 0} (A - C) = \log 2$.

To avoid the coupled boundary condition we make the transformation

$$E = 2B + C - A\tag{A.5}$$

⁶Uniqueness is likely to be violated if that condition is not imposed, as illustrated by the following lower dimensional elliptic example. The infilling Ricci-flat geometries for a manifold with fixed metric on a S^1 boundary are the flat disc but also the cone. This type of non-uniqueness is expected to carry over to the parabolic case.

to find the system

$$e^{4B+2C-2E} \dot{B} = B'' + \frac{1}{r} B' - e^{2B+2C-2E} + \frac{1}{2} r^2 e^{4C-2E} \quad (\text{A.6})$$

$$e^{4B+2C-2E} \dot{C} = C'' + \frac{1}{r} C' - \frac{1}{2} r^2 e^{4C-2E} \quad (\text{A.7})$$

$$e^{4B+2C-2E} \dot{E} = E'' + \frac{1}{r} E' + \left(-\frac{2}{r} E' + \frac{4}{r} B' \right) - \left(C'^2 + 2B'^2 - [2B' + C' - E']^2 \right) + \frac{1}{2} r^2 e^{4C-2E} - 2e^{2B+2C-2E} \quad (\text{A.8})$$

where now $E' = 0$ at the boundary $r = 1$. Next we assume the existence of two even solutions (B, C, E) and $(\tilde{B}, \tilde{C}, \tilde{E})$, both bounded with all derivatives up to order k , say, in $[0, 1] \times [0, T]$ for fixed T . We are going to consider the equations for their difference

$$(\mathcal{B}, \mathcal{C}, \mathcal{E}) = (B - \tilde{B}, C - \tilde{C}, E - \tilde{E}) \quad (\text{A.9})$$

The equations are

$$e^{4B+2C-2E} \dot{\mathcal{B}} = \mathcal{B}'' + \frac{1}{r} \mathcal{B}' + \dot{B} \left(e^{4\tilde{B}+2\tilde{C}-2\tilde{E}} - e^{4B+2C-2E} \right) - \left(e^{2B+2C-2E} - e^{2\tilde{B}+2\tilde{C}-2\tilde{E}} \right) + \frac{1}{2} r^2 \left(e^{4C-2E} - e^{4\tilde{C}-2\tilde{E}} \right) \quad (\text{A.10})$$

$$e^{4B+2C-2E} \dot{\mathcal{C}} = \mathcal{C}'' + \frac{1}{r} \mathcal{C}' + \dot{C} \left(e^{4\tilde{B}+2\tilde{C}-2\tilde{E}} - e^{4B+2C-2E} \right) - \frac{1}{2} r^2 \left(e^{4C-2E} - e^{4\tilde{C}-2\tilde{E}} \right) \quad (\text{A.11})$$

$$e^{4B+2C-2E} \dot{\mathcal{E}} = \mathcal{E}'' + \frac{1}{r} \mathcal{E}' + \left(-\frac{2}{r} \mathcal{E}' + \frac{4}{r} \mathcal{B}' \right) + \dot{E} \left(e^{4\tilde{B}+2\tilde{C}-2\tilde{E}} - e^{4B+2C-2E} \right) - \left(\left[C + \tilde{C} \right]' \mathcal{C}' + 2 \left[B + \tilde{B} \right]' \mathcal{B}' - \left[2B + C - E + 2\tilde{B} + \tilde{C} - \tilde{E} \right]' (2\mathcal{B}' + \mathcal{C}' - \mathcal{E}') \right) + \frac{1}{2} r^2 \left(e^{4C-2E} - e^{4\tilde{C}-2\tilde{E}} \right) - 2 \left(e^{2B+2C-2E} - e^{2\tilde{B}+2\tilde{C}-2\tilde{E}} \right) \quad (\text{A.12})$$

with the initial data

$$\mathcal{B}(0, r) = \mathcal{C}(0, r) = \mathcal{E}(0, r) = 0 \quad (\text{A.13})$$

and boundary conditions

$$\mathcal{B}(t, 1) = \mathcal{C}(t, 1) = 0 \quad \text{and} \quad \mathcal{E}'(t, 1) = 0 \quad (\text{A.14})$$

as well as the conical condition

$$\mathcal{E} - 2\mathcal{B} \sim r^2 \quad \text{at the origin} \quad (\text{A.15})$$

plus regularity conditions at the origin arising from the fact that the functions are even.

We now multiply equation (A.10) by $(r\mathcal{B} + r\dot{\mathcal{B}})$, (A.11) by $r\mathcal{C}$, (A.12) by $\frac{1}{k}r(\mathcal{E} - 2\mathcal{B})$, add the terms up and integrate over time and space. Here k is a fixed constant which we are going to determine below ($k = 16$ will do for instance). On the left hand side we obtain for the \mathcal{B} equation (A.10) when multiplied by $(r\mathcal{B} + r\dot{\mathcal{B}})$ and integrated

$$\begin{aligned} & \int_{t_1}^{t_2} \int_0^1 r \left[e^{4B+2C-2E} (\dot{\mathcal{B}}\mathcal{B} + \dot{\mathcal{B}}^2) \right] dr dt = \\ & \quad \frac{1}{2} \int_0^1 r e^{4B+2C-2E} \mathcal{B}^2 dr \Big|_{t=t_1}^{t=t_2} \\ & \quad + \int_{t_1}^{t_2} \int_0^1 r e^{4B+2C-2E} \dot{\mathcal{B}}^2 dr dt \\ & + \frac{1}{2} \int_{t_1}^{t_2} \int_0^1 r e^{4B+2C-2E} (-4\dot{\mathcal{B}} - 2\dot{\mathcal{C}} + 2\dot{\mathcal{E}}) \mathcal{B}^2 dr dt \end{aligned} \quad (\text{A.16})$$

for the \mathcal{C} equation (A.11) when multiplied by $\mathcal{C}r$ and integrated we find

$$\begin{aligned} & \int_{t_1}^{t_2} \int_0^1 r \left[e^{4B+2C-2E} (\mathcal{C}\partial_t \mathcal{C}) \right] dr dt = \\ & \quad \frac{1}{2} \int_0^1 r e^{4B+2C-2E} \mathcal{C}^2 dr \Big|_{t=t_1}^{t=t_2} \\ & + \frac{1}{2} \int_{t_1}^{t_2} \int_0^1 r e^{4B+2C-2E} \left[(-4\dot{\mathcal{B}} - 2\dot{\mathcal{C}} + 2\dot{\mathcal{E}}) \mathcal{C}^2 \right] dr dt \end{aligned} \quad (\text{A.17})$$

and finally for the left hand side of the \mathcal{E} equation (A.12) after multiplication by $\frac{1}{k}r(\mathcal{E} - 2\mathcal{B})$ and integration

$$\begin{aligned} & \int_{t_1}^{t_2} \int_0^1 \frac{1}{k} r \left[e^{4B+2C-2E} (\partial_t \mathcal{E}) (\mathcal{E} - 2\mathcal{B}) \right] dr dt = \\ & \quad \frac{1}{2k} \int_0^1 r e^{4B+2C-2E} (\mathcal{E}^2 - 4\mathcal{E}\mathcal{B}) dr \Big|_{t=t_1}^{t=t_2} \\ & + \int_{t_1}^{t_2} \int_0^1 \frac{1}{2k} r e^{4B+2C-2E} \left[(-4\dot{\mathcal{B}} - 2\dot{\mathcal{C}} + 2\dot{\mathcal{E}}) (\mathcal{E}^2 - 4\mathcal{E}\mathcal{B}) \right] dr dt \\ & \quad + \int_{t_1}^{t_2} \int_0^1 \frac{1}{k} r \left[e^{4B+2C-2E} (2\mathcal{E}\dot{\mathcal{B}}) \right] dr dt \end{aligned} \quad (\text{A.18})$$

The spacetime integrals of equations (A.16), (A.17), (A.18) not containing a time-derivative of \mathcal{B} , \mathcal{C} or \mathcal{E} are controlled by

$$K \int_{t_1}^{t_2} \int_0^1 r e^{4B+2C-2E} (\mathcal{B}^2 + \mathcal{C}^2 + \mathcal{E}^2) dr dt \quad (\text{A.19})$$

where K is a large constant. This follows from the fact that we control the time-derivatives of the functions A, B and C by some constant. Note also that the term in the last line of (A.18) can be

bounded by

$$\int_{t_1}^{t_2} \int_0^1 \frac{1}{k} r \left[e^{4B+2C-2E} \left(2\mathcal{E}\dot{\mathcal{B}} \right) \right] dr dt \leq \int_{t_1}^{t_2} \int_0^1 \frac{1}{k} r \left[e^{4B+2C-2E} \left(\mathcal{E}^2 + \dot{\mathcal{B}}^2 \right) \right] dr dt \quad (\text{A.20})$$

Let us turn to the terms which are produced on the right hand side when equation (A.10) is multiplied by $(r\mathcal{B} + r\dot{\mathcal{B}})$ followed by a spacetime integration. For the highest order derivative terms we obtain

$$\begin{aligned} \int_{t_1}^{t_2} \int_0^1 \left(\mathcal{B}'' + \frac{1}{r} \mathcal{B}' \right) (\mathcal{B} + \dot{\mathcal{B}}) r dr dt &= \int_{t_1}^{t_2} r \left(\mathcal{B}' \mathcal{B} + \mathcal{B}' \dot{\mathcal{B}} \right) \Big|_{r=0}^{r=1} dt \\ &\quad - \frac{1}{2} \int_0^1 (\mathcal{B}')^2 r dr \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_0^1 r (\mathcal{B}')^2 dr dt \end{aligned} \quad (\text{A.21})$$

The boundary term vanishes because \mathcal{B} satisfies Dirichlet conditions at $r = 1$ and Neumann conditions at $r = 0$ (even function). Analogously we estimate the highest order terms appearing on the right hand side of equation (A.11) when multiplied by $\mathcal{C}r$:

$$\int_{t_1}^{t_2} \int_0^1 \left(\mathcal{C}'' + \frac{1}{r} \mathcal{C}' \right) \mathcal{C} r dr dt = \int_{t_1}^{t_2} r \mathcal{C}' \mathcal{C} \Big|_{r=0}^{r=1} dt - \int_{t_1}^{t_2} \int_0^1 r (\mathcal{C}')^2 dr dt \quad (\text{A.22})$$

where again the boundary term vanishes. Finally, for the right hand side of equation (A.12) being multiplied by $\frac{1}{k} r (\mathcal{E} - 2\mathcal{B})$ we use

$$\begin{aligned} &\int_{t_1}^{t_2} \int_0^1 \frac{1}{k} \left(\mathcal{E}'' + \frac{1}{r} \mathcal{E}' \right) (\mathcal{E} - 2\mathcal{B}) r dr dt + \int_{t_1}^{t_2} \int_0^1 \frac{1}{k} \left(-\frac{2}{r} \mathcal{E}' + \frac{4}{r} \mathcal{B}' \right) (\mathcal{E} - 2\mathcal{B}) r dr dt \\ &= \int_{t_1}^{t_2} \frac{1}{k} r \mathcal{E}' (\mathcal{E} - 2\mathcal{B}) \Big|_{r=0}^{r=1} dt - \int_{t_1}^{t_2} \int_0^1 \frac{1}{k} r \left((\mathcal{E}')^2 - 2\mathcal{B}' \mathcal{E}' \right) dr dt - \int_{t_1}^{t_2} \frac{1}{k} (\mathcal{E} - 2\mathcal{B})^2 \Big|_{r=0}^{r=1} dt \end{aligned} \quad (\text{A.23})$$

The first boundary term vanishes because \mathcal{E} satisfies Neumann conditions at $r = 1$ and manifestly vanishes at $r = 0$. The second boundary term does not vanish but since \mathcal{B} satisfies Dirichlet conditions at $r = 1$ and the conical condition is satisfied at $r = 0$ it equals

$$- \int_{t_1}^{t_2} \frac{1}{k} (\mathcal{E} - 2\mathcal{B})^2 \Big|_{r=0}^{r=1} dt = - \int_{t_1}^{t_2} \frac{1}{k} \mathcal{E}^2(t, 1) dt \leq 0 \quad (\text{A.24})$$

Note that without the conical condition imposed, the sign of this term would not be controlled and the uniqueness proof would fail.

The other terms appearing on the right hand side in the process of the multiplication and integration will be estimated as follows. For the difference of the exponentials we observe

$$\begin{aligned} e^{4B+2C-2E} &- e^{4\tilde{B}+2\tilde{C}-2\tilde{E}} = \int_0^1 \frac{\partial}{\partial \tau} \left[e^{(4B+2C-2E)\tau + (1-\tau)(4\tilde{B}+2\tilde{C}-2\tilde{E})} \right] d\tau \\ &= \left[\int_0^1 e^{(4B+2C-2E)\tau + (1-\tau)(4\tilde{B}+2\tilde{C}-2\tilde{E})} d\tau \right] (4\mathcal{B} + 2\mathcal{C} - 2\mathcal{E}) \\ &\leq C (|\mathcal{B}| + |\mathcal{C}| + |\mathcal{E}|) \end{aligned} \quad (\text{A.25})$$

for some constant C , following from the fact that the solutions (B, C, E) and $(\tilde{B}, \tilde{C}, \tilde{E})$ are bounded. Consequently, the remaining terms on the right hand side of the \mathcal{B} equation when multiplied by $(\mathcal{B}r + \dot{\mathcal{B}}r)$ can be estimated (note that the expression $e^{4B+2C-2E}$ is bounded above and below by some constant)

$$rem_{\mathcal{B}} \leq K \int_{t_1}^{t_2} \int_0^1 r (\mathcal{B}^2 + \mathcal{C}^2 + \mathcal{E}^2) dr dt + \epsilon \int_{t_1}^{t_2} \int_0^1 r e^{4B+2C-2E} (\dot{\mathcal{B}}^2) dr dt. \quad (\text{A.26})$$

Here we frequently applied Cauchy's inequality

$$ab \leq \frac{\nu}{2} a^2 + \frac{1}{2\nu} b^2, \quad (\text{A.27})$$

by means of which the ϵ in (A.26) can be made arbitrarily small on the cost of K becoming larger and larger. By the same token – since the remaining terms of the \mathcal{C} equation are just given by differences of exponentials of the type considered in (A.25) – we can estimate

$$rem_{\mathcal{C}} \leq K \int_{t_1}^{t_2} \int_0^1 r (\mathcal{B}^2 + \mathcal{C}^2 + \mathcal{E}^2) \quad (\text{A.28})$$

For the remainder terms of the \mathcal{E} equation (multiplied by $\frac{1}{k}r(\mathcal{E} - 2\mathcal{B})$) we estimate

$$\begin{aligned} & \left| \frac{1}{k} (\mathcal{E} - 2\mathcal{B}) r \left(\left[C + \tilde{C} \right]' \mathcal{C}' + 2 \left[B + \tilde{B} \right]' \mathcal{B}' \right. \right. \\ & \quad \left. \left. - \left[2B + C - E + 2\tilde{B} + \tilde{C} - \tilde{E} \right]' (2\mathcal{B}' + \mathcal{C}' - \mathcal{E}') \right) \right| \\ & \leq \delta \cdot r (\mathcal{E}'^2 + \mathcal{B}'^2 + \mathcal{C}'^2) + K \cdot r (\mathcal{E}^2 + \mathcal{B}^2 + \mathcal{C}^2) \end{aligned} \quad (\text{A.29})$$

where we have used inequality (A.27) and the fact that the spatial derivatives of the solutions are bounded. Taking the other terms (again differences of exponentials) into account we arrive at the estimate

$$rem_{\mathcal{E}} \leq K \int_{t_1}^{t_2} \int_0^1 r (\mathcal{B}^2 + \mathcal{C}^2 + \mathcal{E}^2) dr dt + \delta \int_{t_1}^{t_2} \int_0^1 r (\mathcal{B}'^2 + \mathcal{C}'^2 + \mathcal{E}'^2) dr dt \quad (\text{A.30})$$

where K is some constant. Adding up the equations we find the inequality

$$\begin{aligned} & \frac{1}{2} \int_0^1 r e^{4B+2C-2E} \left[\mathcal{C}^2 + \frac{1}{2k} \mathcal{E}^2 + \frac{1}{2k} (\mathcal{E} - 4\mathcal{B})^2 + \left(1 - \frac{8}{k} \right) \mathcal{B}^2 \right] dr \Big|_{t=t_1}^{t=t_2} \\ & + \frac{1}{2} \int_0^1 \mathcal{B}'^2 r dr \Big|_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2} \int_0^1 r \left(1 - \frac{1}{k} - \epsilon \right) e^{4B+2C-2E} \dot{\mathcal{B}}^2 dr dt \\ & + \int_{t_1}^{t_2} \int_0^1 r \left(\frac{1}{2k} (\mathcal{E}' - 2\mathcal{B}')^2 + \left(\frac{1}{2k} - \delta \right) (\mathcal{E}')^2 + \left(1 - \frac{2}{k} - \delta \right) \mathcal{B}'^2 + (1 - \delta) \mathcal{C}'^2 \right) dr dt \\ & \leq \hat{K} \int_{t_1}^{t_2} \int_0^1 r (\mathcal{B}^2 + \mathcal{C}^2 + \mathcal{E}^2) dr dt \end{aligned} \quad (\text{A.31})$$

We fix k first (say $k = 16$), then choose ϵ, δ small enough (say $\epsilon = \delta = \frac{1}{64}$) to make all terms on the left hand side of expression (A.31) manifestly non-negative. This fixes the constant \hat{K} because (A.27) has to be applied with a certain weight to produce the chosen ϵ and δ in the estimates (A.26) and (A.29). Since the time-dependent, manifestly non-negative expression

$$\begin{aligned} P(t) &= \frac{1}{2} \int_0^1 r e^{4B+2C-2E} \left[C^2 + \frac{1}{2k} \mathcal{E}^2 + \frac{1}{2k} (\mathcal{E} - 4\mathcal{B})^2 + \left(1 - \frac{8}{k}\right) \mathcal{B}^2 \right] dr \\ &+ \frac{1}{2} \int_0^1 \mathcal{B}'^2 r dr \end{aligned} \quad (\text{A.32})$$

is zero at $t_1 = 0$ we can find an interval $[t_1, \tilde{t}]$ with $\tilde{t} \leq T$ in which expression (A.32) is monotonically increasing or constant in time. We then have

$$\max_{t \in [t_1, t_2]} P(t) \leq \hat{K} \int_{t_1}^{t_2} \int_0^1 r (\mathcal{B}^2 + \mathcal{C}^2 + \mathcal{E}^2) dr dt \leq \tilde{K} \int_{t_1}^{t_2} P(t) dt \leq \tilde{K} (t_2 - t_1) \max_{t \in [t_1, t_2]} P(t) \quad (\text{A.33})$$

for some constant \tilde{K} . Inequality (A.33) is valid for all $t_2 \leq \tilde{t}$. Finally, decomposing the time-interval $[t_1, \tilde{t}]$ into a finite number of subintervals, each of size $\tilde{K} \Delta t = \frac{1}{2}$, we find that $P(t) = 0$ for each subinterval and since $P(t)$ is manifestly non-negative, the solution has to be zero on each such subinterval. But if \tilde{t} was not T , expression (A.32) must be strictly monotonically decreasing on a following small interval $[\tilde{t}, \tilde{t} + \epsilon]$, which is impossible since it was just shown that the manifestly non-negative expression (A.32) is zero at $t = \tilde{t}$. Hence $\tilde{t} = T$ and $\mathcal{B} = \mathcal{C} = \mathcal{E} = 0$ everywhere on $[0, 1] \times [0, T]$.

We note that a similar calculation can be performed for the equations following from the NUT-ansatz

$$g = e^{2A(t,r)} dr^2 + r^2 e^{2B(t,r)} (\sigma_1^2 + \sigma_2^2) + r^2 e^{2C(t,r)} \sigma_3^2 \quad (\text{A.34})$$

with the gauge

$$\xi = \left[- (2B'(t, r) + C'(t, r) - A'(t, r)) + (2B'_0(r) + C'_0(r) - A'_0(r)) \right] dr \quad (\text{A.35})$$

where $A_0(r)$, $B_0(r)$ and $C_0(r)$ are the functions chosen to obtain the Ricci-flat Taub-NUT metric in (A.34).

References

- [1] B. Chow, D. Knopf, *The Ricci-Flow: An Introduction*, AMS Surv. a. Monogr., Vol 110 (2004)
- [2] D. M. DeTurck, *J. Differential Geom.* **18** (1983), no. 1, 157–162
- [3] B. Fornberg, *SIAM J. Sci. Comput.* **16** (1995) no. 5, 1071–1081
- [4] G. W. Gibbons and S. W. Hawking, *Commun. Math. Phys.* **66** (1979) 291.
- [5] G. W. Gibbons, S. W. Hawking and M. J. Perry, *Nucl. Phys. B* **138** (1978) 141.
- [6] R. S. Hamilton, *J. Differential Geom.* **17** (1982), no. 2, 255–306.

- [7] R. S. Hamilton, *Surv. Differential Geom.* (1995), vol 2, 7–136.
- [8] M. Headrick and T. Wiseman, *Class. Quant. Grav.* **23** (2006) 6683 [arXiv:hep-th/0606086].
- [9] O. A. Ladyženskaja, V. A. Solonnikov, N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monographs **23**, Providence 1968
- [10] G. Perelman, arXiv:math.dg/0211159.
- [11] G. Perelman, arXiv:math.dg/0303109.
- [12] Y. Shen, *Pacific J. Math.* **173** (1996), no. 1, 203–221.
- [13] L. N. Trefethen, *Spectral methods in MATLAB*, SIAM, Philadelphia, 2000
- [14] C. Warnick, *Class. Quant. Grav.* **23** (2006) 3801 [arXiv:hep-th/0602127].
- [15] R. E. Young, *Phys. Rev. D* **28** (1983) 2420.